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A Mathematical Model for Water Hammer Problem due to Transient Pressure in a Water Pipe Line

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ABSTRACT: Water hammer is high pressure wave generated by sudden changes of velocity in closed pipe line. In this paper an attempt is made to solve water hammer equations due to transient pressure by numerical method and analytical method both. We have made some assumptions to obtain the analytical solution of nonlinear water hammer equations without deviating from the fundamental nature of equations. Transient pressure and flow rate at different location of pipe are calculated with the help of Matlab for both the methods.

Keyword: Water hammer, transient Pressure, turbulent flow, column separation, viscoelasticity

INTRODUCTION

We can experience water hammer (however small) in the domestic water pipeline also. Many times as we close the water tap very quickly we feel the vibrations in the entire pipe and can listen a heavy knocking sound. This is due to rapid deceleration of water (i.e. change in velocity) in the pipe line when the water tap is closed quickly. The same behavior can be seen in a pumping station due to very fast shut down of valve. Many times the surge pressure is as much high that it can damage or facture the pipes at one or more places in the pipe line system. Gottlieb *et al*. [7] investigated numerical model with experimental result. They considered four different configurations of an steel and a plastic pipeline. Extremely high-pressure peaks were recorded immediately upon collapse of the vapor cavity. The pressure then dropped to about 40% of the pressure peak level and maintained this level for twice of l/a seconds. They showed the presence of peaks resembled the pressure peaks associated with the implosion of gas bubbles in pumps. Goldberg [6] presented solution of broad cross section of time varying wave problems in hydraulics, MOC based methods are likely to continue to see extension use. They showed extent of potential application of the implicit time-line method also deserves more attention. Martin [11] presented transient cavitations in as reservoir-pipe-valve system. The water contained a minimal amount of dissolved gas. They emphasized on limited cavitations. They showed in experimental results that the maximum pressure may exceed the Joukowsky pressure rise in the form of short duration pressure pulse. They observed that the reservoir pressure was rising during the experiment because the tank was to small. Shimada *et al.* [12] solve equation using series solution method and compared the result with finite difference method Simpson [14] showed a range of short duration pressure pulses measured in a reservoir upward sloping pipeline valve system. Due to upward slope of the pipe vapor cavity was confined to be adjacent to the valve with no distributed cavitations along the pipe. Sibetheros *et al.* [13] investigated the method of characteristics with spline polynomials for interpolations for numerical water hammer analysis for a frictionless horizontal pipe. Brunone *et al.* [3] developed 2-D model and considered rapid damping of the pressure peaks after the end of a complete closure maneuver, are closely linked to the shape of the cross-sectional velocity distributions and their Variability in time. Ghidaoui *et al.* [5] performed linear stability analysis of base flow velocity profiles for laminar and turbulent water hammer flows. The base flow velocity profile is determined analytically. Where the transient is generated by instantaneous reduction in flow rate at the downstream end of a single pipe system .The presence of inflection points in the base flow velocity profile and the large velocity gradient near the pipe wall are the sources of flow instability. Ghidaoui [4] reviewed the relation between straight equation and speeds in single as well as two phase transient flow by using order of magnitude analysis. Bergant *et al.* [2] showed that unsteady friction, cavitations including Column Separation and trapped air pockets, fluid structure interaction, pipe wall viscoelasticity and leakages and blockages in transient pipe flow. Their models are based on method of characteristics. Yeh *et al.* [16] described the attenuation of water hammer pressure wave with time varying valve closure by using an asymptotic analysis. They examined the effect of flow reversal on the pressure wave attenuation through comparison with a similar method applied to the water hammer generated during flow establishment the flow reversal. Kumar and Pratap [9] suggested that at short wavelength limits the secular equation reduces to Rayleight surface wave frequency equation. Kepler [8] studied hydrolic transient effect on poly vinyle pipe.

They computed attenuation coefficients analytically and dispersion curves graphically. Willey *et al.* [15] used MOC with the limitation of Couraents stability conditions in water hammer simulation and predicated that the coupled method is effective in simulating water hammer in pipe lines. In this paper one dimensional case of water flow in a horizontal pipe with uniform velocity over the cross section of the pipe is considered.

MATHEMATICAL MODEL

Assuming that the wall of pipe is linearly elastic and obeys the law that stress is proportional to strain. The governing equations of motions are

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+g\frac{\partial H}{\partial x}+\frac{fU^2}{2D}=0 \quad (1)$$

In terms of fluid discharge Q it can be written as

$$\frac{\partial Q}{\partial t}+Q\frac{\partial Q}{\partial x}+g\frac{\partial H}{\partial x}+\frac{fQ^2}{2DA}=0 \quad (2)$$

In consideration of reverse flow it becomes-

$$\frac{\partial Q}{\partial t}+Q\frac{\partial Q}{\partial x}+g\frac{\partial H}{\partial x}+\frac{fQ|Q|}{2DA}=0 \quad (3)$$

where f is friction factor and D is diameter of pipe. The equation of continuity is

$$\frac{1}{gA}\frac{\partial Q}{\partial x}+\rho\left(\frac{1+\frac{KD}{\rho E}}{K}\right)\frac{\partial H}{\partial t}=0 \quad (4)$$

where $Q=UA$, $p=\rho gH$ and K is bulk modulus of elasticity of the material of pipe wall.

If a be the velocity of water hammer then equation of continuity for water hammer is

$$\frac{\partial H}{\partial t}+\frac{a^2}{gA}\frac{\partial Q}{\partial x}=0 \quad (5)$$

where $a^2=\dfrac{K}{\rho\left(1+\dfrac{KD}{\rho E}\right)}$

The initial and boundary conditions are

$$H(x,0)=H_0,\ \left(\frac{\partial H}{\partial t}\right)_{t=0}=0 \quad (6)$$

$$Q(x,0)=Q_0,\ \left(\frac{\partial Q}{\partial t}\right)_{t=0}=0 \quad (7)$$

Assuming linear law for valve closing

$$Q(L,t)=\left(1-\frac{t}{t_c}\right)Q_0 \quad (8)$$

Consider the nonlinear term $\dfrac{f}{2DA}Q|Q|$ and horizontal pipe line system. The nonlinear system of water hammer equations under the above assumptions can be written as

$$gA\frac{\partial H}{\partial t}=-a^2\frac{\partial Q}{\partial x} \quad (9)$$

and equation of motion becomes

$$\frac{\partial Q}{\partial t}+gA\frac{\partial H}{\partial x}+\frac{f}{2DA}Q|Q|=0 \quad (10)$$

On differentiating partially equation (9) with respect to x and equation (10) with respect to t partially respectively and then subtracting them, we get

$$\frac{\partial^2 Q}{\partial t^2}+\frac{f}{2DA}\left[|Q|\frac{\partial Q}{\partial t}+Q\frac{\partial |Q|}{\partial t}\right]-a^2\frac{\partial^2 Q}{\partial x^2}=0 \quad (11)$$

On putting the value $\dfrac{\partial |Q|}{\partial t}=\dfrac{Q}{|Q|}\dfrac{\partial Q}{\partial t}$ and simplifying we get

$$\frac{\partial^2 Q}{\partial t^2}+a_1|Q|\frac{\partial Q}{\partial t}=a^2\frac{\partial^2 Q}{\partial x^2} \quad (12)$$

where $a_1=\dfrac{f}{DA}$, f is friction factor, D is pipe diameter and A is the area of cross section of pipe.

METHODS OF SOLUTION

A. Numerical Method

Applying mixed difference scheme i.e. explicit central difference for second order derivative and backward difference for first order derivative as

$$\frac{\partial^2 Q}{\partial t^2}=\frac{Q_{i,k+1}-2Q_{i,k}+Q_{i,k-1}}{\Delta t^2} \quad (13)$$

$$\frac{\partial^2 Q}{\partial x^2}=\frac{Q_{i+1,k}-2Q_{i,k}+Q_{i-1,k}}{\Delta x^2} \quad (14)$$

$$\frac{\partial Q}{\partial t}=\frac{Q_{i,k}-Q_{i,k-1}}{\Delta t} \quad (15)$$

Substituting these values in equation (12), we get

$$\frac{Q_{i,k+1}-2Q_{i,k}+Q_{i,k-1}}{\Delta t^2}+a_1\left[\left|Q_{i,k}\right|\frac{Q_{i,k}-Q_{i,k-1}}{\Delta t}\right]-a^2\left[\frac{Q_{i+1,k}-2Q_{i,k}+Q_{i-1,k}}{\Delta x^2}\right]=0 \tag{16}$$

$$Q_{i,k+1}=r(Q_{i+1,k}+Q_{i-1,k})+2(1-r)Q_{i,k}-Q_{i,k-1}-r_1\left|Q_{i,k}\right|\left(Q_{i,k}-Q_{i,k-1}\right) \tag{17}$$

where $r=\left(\frac{a\Delta t}{\Delta x}\right)^2$, $r_1=\frac{a_1\Delta t}{2}$

The initial conditions are

$Q(x,0)=Q_0$ (initial flow in the pipe), (18a)

$$\left(\frac{\partial Q}{\partial t}\right)_{t=0}=0 \tag{18b}$$

$$H(x,0)=H_0,\ \left(\frac{\partial H}{\partial t}\right)_{t=0}=0 \tag{19}$$

The boundary conditions are

$$Q(0,t)=Q_0 \tag{20}$$

$$Q(L,t)=Q_0\left(1-\frac{t}{t_c}\right) \tag{21}$$

where t_c is valve closure time.

Similarly the equation of continuity becomes

$$H_{i,k+1}=H_{i,k-1}-r_2\left(Q_{i+1,k}-Q_{i-1,k}\right) \tag{22}$$

where $r_2=\frac{-a^2\Delta t}{gA\Delta x}$

B. Analytical Method

To obtain analytical solution let us convert this equation into another form by using co-ordinate transformations

$$\tau=a_1t\ ,\ z=\frac{a_1x}{a} \tag{23}$$

in equation (12), we get

$$\frac{\partial^2Q}{\partial\tau^2}+|Q|\frac{\partial Q}{\partial\tau}=\frac{\partial^2Q}{\partial z^2} \tag{24}$$

The analytical solution of equation (24) is obtained by using the transformation techniques (Andrei Polyanin and Valentin [1]) and is given by

$$Q=\pm2c_1\tanh\left(\alpha_1x-\alpha_2t+c_2\right) \tag{25}$$

where $\alpha_1=\frac{a_1A_1}{\left(1-A_1^{\,2}\right)a}$, $\alpha_2=\frac{a_1A_1^{\,2}}{\left(1-A_1^{\,2}\right)}$, c_1, c_2 and A_1 are arbitrary constants with the condition that $A_1\neq\pm1,0$.

Applying the initial and the boundary conditions, we get

$c_2=-\alpha_1L+\alpha_2t_c$ and

$$c_1=\frac{Q_0}{2\tanh\left[\alpha_1\left(x-L\right)+\alpha_2t_c\right]} \tag{26}$$

Substituting the value of Q in equation (24) and solving together with initial condition $H(x,0)=H_0$, we get

$$H=\frac{2\alpha_1a^2\left(1-A_1^2\right)}{a_1gAA_1^2}c_1\left[\tanh\left(\alpha_1x-\alpha_2t+c_2\right)-\tanh\left(\alpha_1x+c_2\right)\right]+H_0 \tag{27}$$

where α_1, α_2, c_1, c_2 are same as given above. We need one more boundary condition to calculate the value of constant A_1.

RESULTS AND DISCUSSION

To understand the behavior of pressure head H and flow rate Q, we have used MATLAB to plot the graphs for the model. The initial value of pressure head H_0 =600m initial flow Q_0 =1.4m^3/s, length of pipe L =100 m and wave velocity c=1000 m/s and pipe radius r = 0.3m. The numerical values for flow Q and pressure H are obtained from the difference equations with the help of MATLAB and are shown graphically from figure 1 to figure 2. In figure 1 the variation of water flow in the pipe together with axial distance x and time t is shown. Valve closure time is taken 0.10 s .The another sample parameter are taken as wave velocity c=1000 m/s, pipe radius r is 0.3 m, initial pressure head H_0 = 600 m and initial flow Q_0 = 1.4m^3/s. The variation of flow with time and distance can be clearly observed form figure 1. Initially the variation of flow with x is very small and at t = 0.1 s and x = 100 it becomes zero but it changes rapidly as time increases. The value of flow at some middle points in the pipe is larger than the initial value Q_0. This is feasible since as the valve is going to close then there is some back flow causing greater values of Q at mixing points. As time exceeds the value of flow rate increases in the upstream direction (i.e. negative value of Q) which is true since the valve is closed and there is no chance for downstream flow.

The variation of pressure head with respect to time and axial distance is shown in figure 2. The pressure head in the pipe increases as time increases. We have observed that at $t = t_c$ there is negative transient pressure head which may cause cavitations or column separation. After that transient pressure head increases as time increases and is maximum at $\frac{t}{T} = 1$ i.e. $t = T = 2s$. We observed $H_{\max} = 7800\,m$. The exact analytical solution is not possible as there is one more arbitrary constant in the solutions A_1 for which there is need of one more condition. In the above solution there is restriction that $A_1 \neq \pm 1$, to draw the graph for the analytical solution we have taken $A_1 = 2.5$ and result are shown from the figures 3. It is clear from the figure that pressure head increases as time increases and it is maximum at the valve. The comparison of pressure head obtain by numerical method and analytical method for nonlinear model is shown in figure 4, and the results are in good agreement.

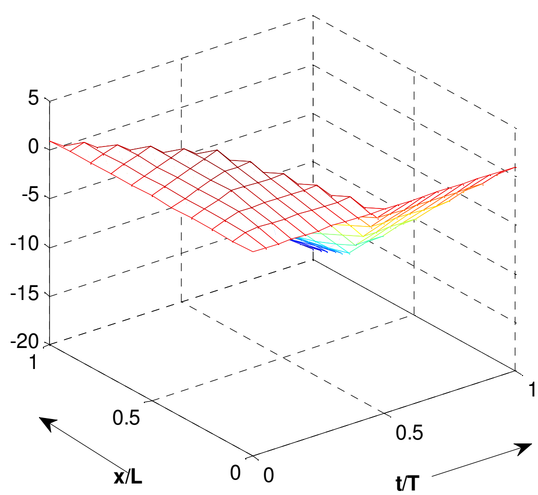

Fig. 1. Variation of Flow in Pipe with x and t.

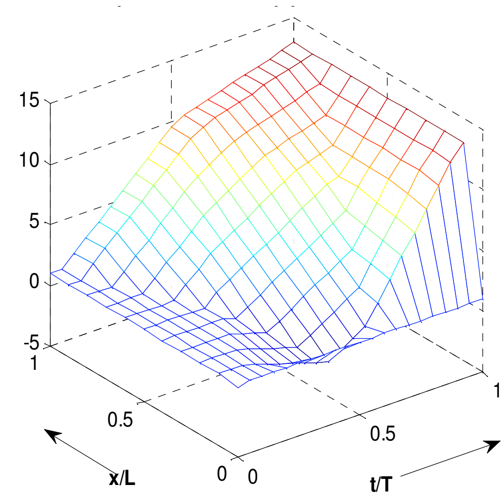

Fig. 2. Variation of pressure in pipe with x and t.

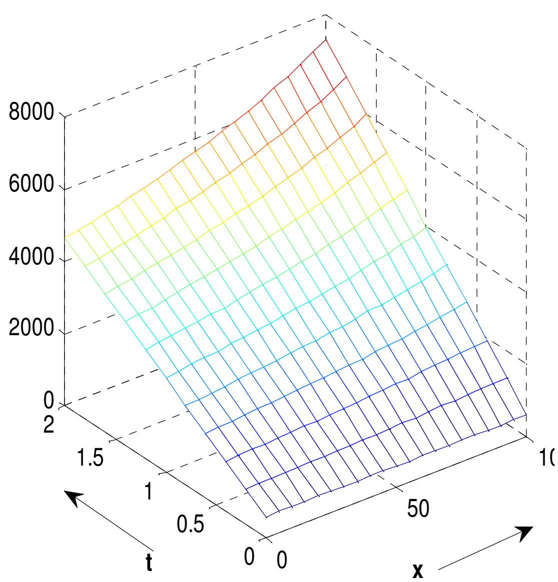

Fig. 3. Variation of pressure due to closure of valve.

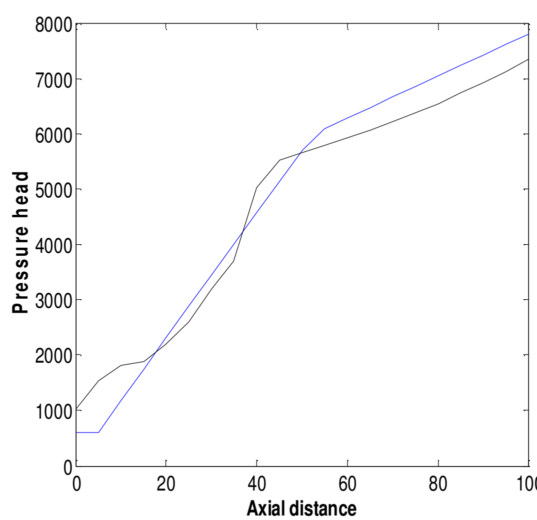

Fig. 4. Comparison of pressure head by numerical and analytical method.

CONCLUSION

An analytical as well as a numerical solution for linear and nonlinear systems of water hammer equations is discussed in this model. It is observed that transient pressure in pipe line increases very fast, when valve is closed suddenly. The transient pressure increases as the valve closure time decreases the negative flow and negative transient pressure is also observed at some points in the pipe which may cause column separation.

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